

FINITE TYPE INVARIANTS OF LINKS WITH A FIXED LINKING MATRIX

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ABSTRACT. In this paper we introduce two theories of finite type invariants for framed links with a fixed linking matrix. We show that these theories are different from, but related to, the theory of Vassiliev invariants of knots and links. We will take special note of the case of zero linking matrix. i.e., zero-framed algebraically split links. We also study the corresponding spaces of “chord diagrams”.

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1. INTRODUCTION

We divide the set of all framed link types (isotopy classes of framed links in S^3) into equivalence classes according to their linking matrices. Namely, every equivalence class consists of all framed link types having a certain given linking matrix.

There exist for sometime now, definitions of finite type invariants of topological objects such as knots, 3-manifolds, graphs etc. In order to make these definitions, one should have a notion of adjacency. Then, based on this notion it is possible to set conditions for an invariant of a certain type of objects (knots, 3-manifolds etc.) to be of finite type. Recall the

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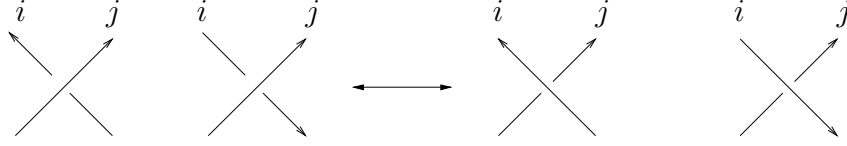


Figure 1. A DD-crossing change

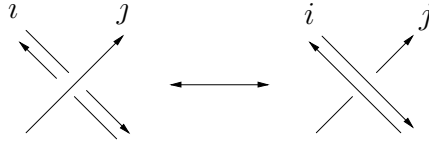


Figure 2. A \wedge -crossing change

standard definition for links (see e.g. [BN1, Bi, BiLi, Go1, Go2, Ko, Va1, Va2] and [BN3]). Two links are adjacent if they differ by a single crossing change. The definition of Vassiliev invariants of links is based on this notion of adjacency. However, if one wishes to set up a finite type theory for links within a given linking class, this notion of adjacency will not do since one crossing change changes some linking number and therefore alters the linking matrix.

In this paper we introduce two forms of finite type invariants of links in a fixed linking class, based on similar, yet slightly different, notions of adjacency.

As a special case we will be interested in zero-framed algebraically split links (zero framed links having zero linking matrix). Such links appear in a very essential form in the theory of finite type invariants of 3-manifolds.¹

Definition 1.1. 1. *We will say that two links in the same linking class are DD-adjacent if they differ by a double crossing change, as indicated in Figure 1, showing two crossings involving the i^{th} and j^{th} components in such a way that their linking number remains unchanged.*

2. *The other type of the adjacency we present here is influenced by the way the crossings to be changed are placed along the link. We will call the two crossings involved “close” if there is a segment of the i^{th} component, say, going from one of the crossings to the other one, meeting no other crossings along the way. This type of crossing change can be visualized as in Figure 2. In this case the links are called \wedge -adjacent.*

Notice that since on the i^{th} component the crossings are close, we can think of them as appearing almost at the same point. This will be of significance while considering the corresponding chord diagrams in this case.

As an intermediate step, for both kinds of crossing changes, we get a link with two double points. Such a link is the basic element of the singular objects in the theory. Still, in order to be able to extend link invariants to singular objects and to set a finiteness condition we need to be more precise in the definition of a singular link in a linking class. Let \mathcal{P} denote the class of links with some fixed linking matrix P .

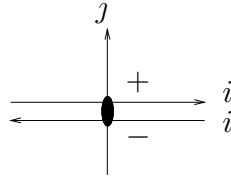
¹To be precise, the links that appear in the theory of finite type invariants of 3-manifolds are unit framed. The difference between unit-framed and zero-framed plays no role in the statements we make in this paper.

Definition 1.2. 1. A DD^m -singular P -link is a link having $2m$ double points paired into m ordered (by $+/-$ signs) pairs, so that each pair is of the form

$$\left(\begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \times \\ \diagdown \quad \diagup \\ j \quad i \end{array} \right) + , \left(\begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \times \\ \diagdown \quad \diagup \\ j \quad i \end{array} \right) -$$

where i and j are components of the link, and so that resolving those pairs in the way described below ends with a link in the class \mathcal{P} .

2. A \bigwedge^m -singular P -link is the same as in 1 but with the exception that all the pairs of double points taken in account are close pairs, as shown in the figure below.



Resolving a pair of singular points is done, in both types of singularity by,

1. For DD -singularity,

$$\left(\begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \times \\ \diagdown \quad \diagup \\ j \quad i \end{array} \right) + , \left(\begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \times \\ \diagdown \quad \diagup \\ j \quad i \end{array} \right) - \longrightarrow \left(\begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ j \quad i \end{array} \right) - \left(\begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ j \quad i \end{array} \right) -$$

$\underline{L} \qquad \qquad \qquad \underline{\bar{L}} \qquad \qquad \qquad \underline{L}$

2. For \bigwedge -singularity,

$$\left(\begin{array}{c} \uparrow \\ \bullet \\ \leftarrow \quad \rightarrow \\ \bullet \\ \uparrow \end{array} \right) \longrightarrow \left(\begin{array}{c} \uparrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \uparrow \end{array} \right) - \left(\begin{array}{c} \uparrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \uparrow \end{array} \right)$$

$\underline{L} \qquad \qquad \qquad \underline{\bar{L}} \qquad \qquad \qquad \underline{L}$

Definition 1.3. If V is a G -valued link invariant where G is any commutative group then we extend V to singular links, in both types by,

$$(1.1) \qquad V(L) = V(\underline{\bar{L}}) - V(\underline{L})$$

Definition 1.4. We say that V is a DD -linking class- P invariant of type m (\bigwedge -invariant of type m) if it vanishes on all DD^m -singular P -links (all \bigwedge^m -singular P -links). We say that V is a $DD(\bigwedge)$ -linking class- P invariant of finite type if it is of type m for some positive m .

A natural question we can ask is how this theory relates to the theory of Vassiliev invariants.

Theorem 1.5. Every Vassiliev invariant is a linking class finite type invariant for all classes.

Proof. The theorem follows from the equality

$$\begin{aligned} V \left(\begin{array}{c} \diagup \quad \diagdown \\ \times \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \times \\ \diagdown \quad \diagup \end{array} \right) &= V \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right) - V \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \\ &= V \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right) + V \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} , \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} \right) \end{aligned}$$

Hence the value of V on a $DD(\bigwedge)$ -singular P-link having $m+1$ pairs of double points descends to the sum of values of V on singular links each of which has $m+1$ double points. \square

Theorem 1.6. *If V is a Vassiliev invariant of pure tangles which descends to a well defined invariant of links within a given linking class, via the closure map, then V is an invariant of type m , in the sense presented here, for the links in that class.*

The proof of this theorem uses exactly the same argument as the previous proof, but before we calculate the value of V on some link we first cut it open to a pure tangle and take the value of V on this tangle.

Corollary 1.7. *Since the Milnor μ_{ijk} is an invariant of pure tangles which is well defined as an invariant of algebraically split links, and by [BN2, Li] is known to be an invariant of type 2, we obtain as a corollary that it is an invariant of at most type 2 in our sense too.*

Theorem 1.8. *(proof in Section 4) The only invariants of finite type for links up to equivalence class in the sense presented here, up to type 2, are restrictions of Vassiliev invariants and μ_{ijk} .*

The paper is organized as follows. In Section 2 we introduce the spaces \mathcal{D}^{DD} of Double Dating diagrams and \mathcal{D}^\wedge of Wedge diagrams and we show that these spaces are the appropriate analogues of chord diagrams. In Section 3 we introduce some relations which are satisfied by double dating and wedge diagrams. In Section 4 we introduce the spaces \mathcal{F}_P^m of all type m invariants, and \mathcal{A}^{DD} which is the quotient space of \mathcal{D}^{DD} divided by the subspace generated by all relations we present in Section 3. Furthermore in Section 4, we calculate the dimension of those spaces up to degree 2, with respect to the number of components. In Section 5 we introduce a map from \mathcal{D}^{DD} to the algebra of chord diagram, from Vassiliev invariants, and we describe the image under this map of \mathcal{D}^{DD} , \mathcal{D}^\wedge . Finally, we present some open questions.

At the time that this preprint was still in preparation I received a preprint [Me] by B. Mellor, which is based on an earlier version of my preprint. Thus there is some overlap between the two works. The basic definitions are nearly the same, and in both places it is shown that the Milnor invariant μ_{ijk} is an invariant of finite type for algebraically split links. The proofs are different though.

1.1. Acknowledgement. The theory presented here is a joint work with Dror Bar-Natan from the Hebrew University in Jerusalem. I am deeply grateful to Dror for his sincere help and advisory during the time this work was done and all along my Ph.D studies.

I also wish to express my deep gratitude to Stavros Garoufalidis for his warm hospitality and the stimulating atmosphere and useful conversations I had with him while visiting him at Harvard University during December 98.

2. THE DIAGRAMS

We wish to present here the objects appropriate to play the role of chord diagrams in each of the two types of iteration processes we have described above. First, we will treat the case of DD-adjacent links, in which Double Dating diagrams (DD-diagrams, for short) will be presented. Then we tend to the case of \bigwedge -adjacent links, and introduce Wedge diagrams

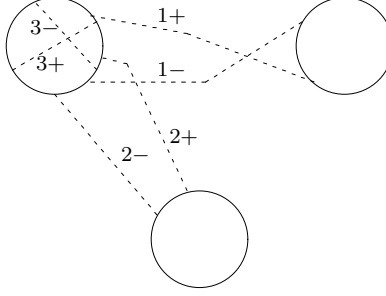


Figure 3. A double dating diagram

2.1. Double Dating Diagrams.

Definition 2.1. A degree m double dating diagram on l components is a collection of l ordered oriented circles, and m ordered pairs, ordered by $+/-$ signs, of dashed lines(chords) so that both chords in each pair have their end points on the same one/two circles. In Figure 3 an example of a double dating diagram of degree 3 on three components is shown. Note that the numerals 1-3 written next to the chords, in the figure, are not part of the structure of a DD-diagrams. They are only a mean of presenting the pairing of the chords.

Let \mathcal{D}_m^{DD} be the span of all double dating diagrams of degree m and let \mathcal{D}^{DD} be the direct sum over all m of \mathcal{D}_m^{DD} .

Theorem 2.2. The space \mathcal{D}_m^{DD} is analogous to the space of chord diagrams of degree m , defined in the theory of Vassiliev invariants. To be more specific,

1. To every degree m DD-singular link, L , there corresponds a degree m DD diagram, D_L .
2. (a) To every degree m DD diagram, D , there exists a degree m singular link, $L(D)$, (in every linking class) so that

$$(2.2) \quad D_{L(D)} = D$$

- (b) Two such links, within the same class, corresponding to the same DD diagram, differ by finite sequence of double crossing changes.

Proof. The proof of 1 is best illustrated by an example shown in Figure 4, where we show a singular link on the left hand side and its corresponding double dating diagram on the right hand side. For the proof of 2a we show how to build a singular link for a given DD diagram. Let D be a DD diagram with some full lines and pairs of chords. We will call the chords belonging to a certain pair “pals”. In Figure 5 we see a double dating diagram on the left hand side. Use “finger moves” along the chords in order to get from (a) to (b). Following this procedure to all pairs of pals chords will give a singular link as desired. One may think of the finger move as axon connecting two neurons by a synapse. Clearly this link is not unique and it can be obtained in any linking class. However now we show that any two singular links respecting the same DD diagram and within the same class differ by a finite sequence of double crossing flips. This is what is claimed in 2b. We say that a link (singular or not), is in a neural position if it is embedded in S^3 so that each component is an unknotted circle remote from all other components, knotted and linked to other components by a network of axons and synapses. Every link is isotopic to a link in a neural position. To see this one can take a disk neighborhood of every crossing of a given link diagram. The crossing can be

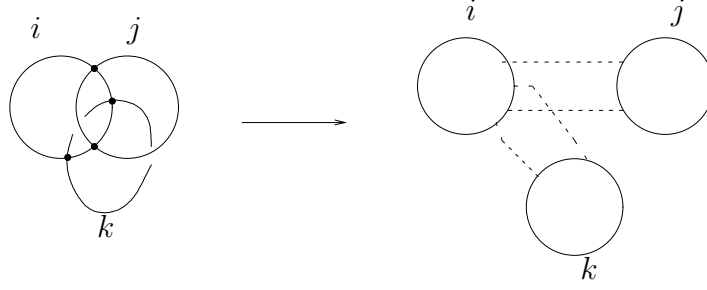


Figure 4. A singular link and its DD-diagram

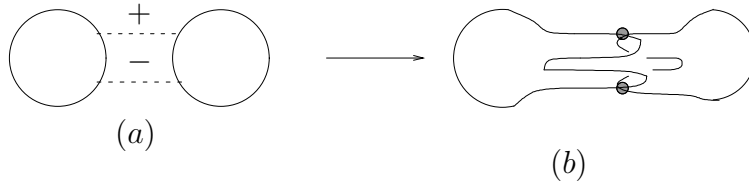


Figure 5. From a diagram to a singular link

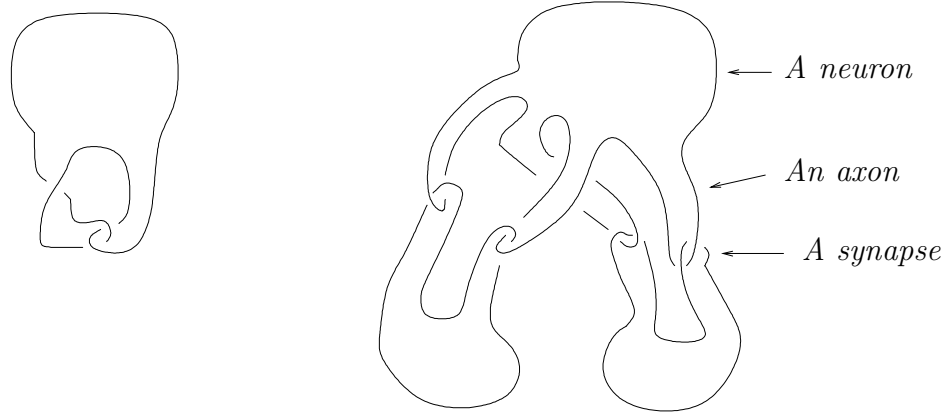


Figure 6. A neuro trefoil knot and a neuro Borromean link

regular or singular. Making the disk thicker one gets a 3-ball neighborhood of the crossing. It is possible to move this ball-neighborhood around in 3-space thus creating an axon of the neuro-link and the crossing neighborhood becomes the synapse. A self intersection point belonging to a singular link will give a singular synapse. A non singular synapse, is either positive or negative according to its contribution (+1 for positive, -1 for negative) to the linking matrix. See Figure 6 for examples of a link and a knot in neural positions, and Figure 7 for the distinguishing between the signs of non-singular synapse and also for an example of a singular synapse. Given two links, L_1, L_2 , respecting the same DD diagram, D and the same linking matrix, we first bring both of them to neural positions. Suppose i_1, i_2, j_1, j_2 are the i^{th} and j^{th} components of L_1, L_2 respectively. The numbers i and j of both links are given by the diagram D . First, we will make sure that modulo double crossing change, both i_1, j_1 and i_2, j_2 have the same number of synapse involving each pair. Since the links respect the same diagram we do not have to worry about singular synapses. The

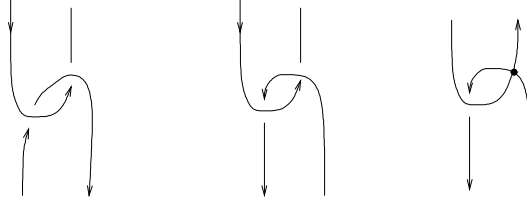
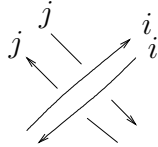


Figure 7. A positive synapse, a negative synapse and a singular synapse (in order)

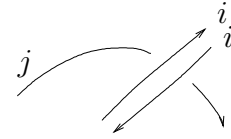
number of them is given from the diagram. suppose that the pair i_2, j_2 has more non-singular synapses than i_1, j_1 . Since the links L_1, L_2 have the same linking matrix, and the fact that each synapse adds ± 1 to the linking matrix we have that,

$$\#\{\text{synapses between } i_2, j_2\} - \#\{\text{synapses between } i_1, j_1\} = \text{an even number}$$

It follows that we can find a set of synapses involving i_2, j_2 representing this difference which can be divided into pairs, each of them consists of one positive and one negative synapse. Every such pair gives us a double crossing flip that allows us to cancel those synapses and hence decreases the number of synapses between i_2, j_2 by 2. We repeat this argument for every pair of components of L_2 . This way we change L_2 into a link, L'_2 , every two components of which has the same number of synapses in the corresponding pair of components in L_1 . In order to finish the proof we still have to show that we can find a sequence of double flips making L'_2 The same as L_1 . Since an axon is made of two strings going in two opposite directions it is transparent to other axons and circles in the sense that we can flip it through axons and circles using only double crossing flips. Therefore the combinatorics of the setting which is defined entirely by the diagram determines the topology.



an axon of the i^{th} component overcrosses an axon of the j^{th} component



an axon of the i^{th} component overcrosses the j^{th} circle component

The links L_1 and L'_2 have the same combinatorial setting and the axons of L_1 can be moved freely, (up to double crossing flips), in space onto the axons of L'_2 so they become isotopic links. This completes the proof. \square

2.2. Wedge Diagrams.

Definition 2.3. A wedge diagram of degree m on l components is a collection of l oriented ordered circles and m dashed wedges, so that the tip of a wedge lies on one circle and both its legs lies on another (maybe the same) circle. The legs of each wedge are ordered by $+/-$ signs. See Figure 8, for an example of a Wedge diagram of degree 3 and 2 components.

Define \mathcal{D}_m^\wedge and \mathcal{D}^\wedge in the same way as \mathcal{D}^{DD} and \mathcal{D}_m^{DD}

Theorem 2.4. The space of chord diagrams corresponding to \wedge -singular links (within a linking class), in the sense of theorem 2.2, is the space \mathcal{D}^\wedge .

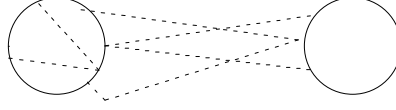
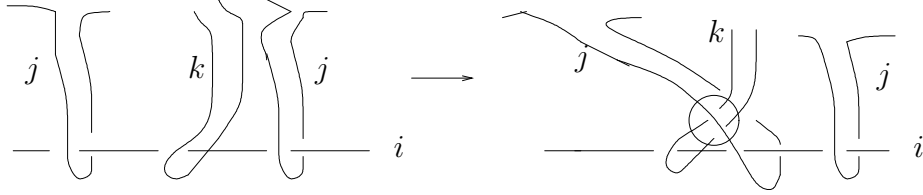


Figure 8. A \wedge -diagram

Proof. The proof of this theorem is basically identical to that of 2.2. The only difference occurs when we reduce the number of axons in L_2 so that it would be the same as for L_1 . as already mentioned there, we cancel a pair of axons by the use of double crossing flips. In general these crossings do not have to be close. If the two axons are adjacent then those crossings are close. If they are not and we have two axons separated by a third one, as below, then we slide one of them as indicated and make the double crossing flip encircled which is a wedge type flip, so that they become adjacent.



□

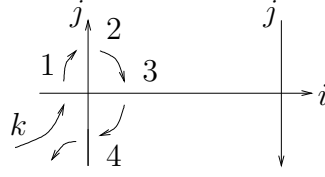
Wedge diagrams naturally mapped into Double Dating diagrams as we can put the end points at the tip of each wedge further from each other. This is actually the illustration of the inclusion map from \mathcal{D}^\wedge into \mathcal{D}^{DD} . At the end of the next section we will show that this map is onto.

3. RELATIONS FOR DIAGRAMS

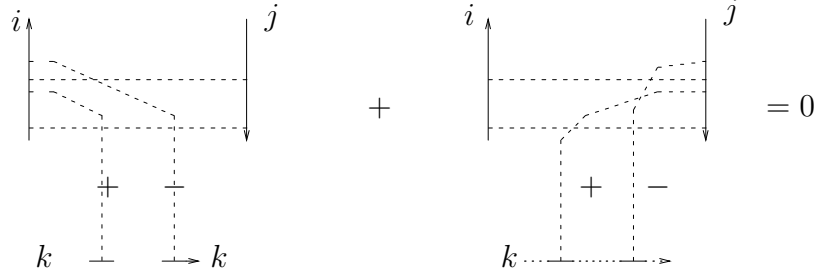
In order to define weight systems in this theory, as well as building a Hutching's theory for giving integration conditions of weight systems to invariants [Hu], we need to have complete knowledge of the relations satisfied by double dating and wedge diagrams. In this section we will present the relations we already know. We suspect that the list of relations below is complete, but we have not proven that yet. Recall from [BN1] that a weight system is a linear functional on the space freely generated by chord diagrams which corresponds to finite type invariants of links. In order to be a weight system oppose to a general linear functional it is necessary that the functional will satisfy certain relations originated by the topological nature of the setting. In the following description of relations we present the topological origin and translate it into relations among DD-diagrams (\wedge -diagrams). A weight system then will become a linear functional on diagrams which vanishes on the subspace spanned by all the relations.

3.1. DD Diagrams.

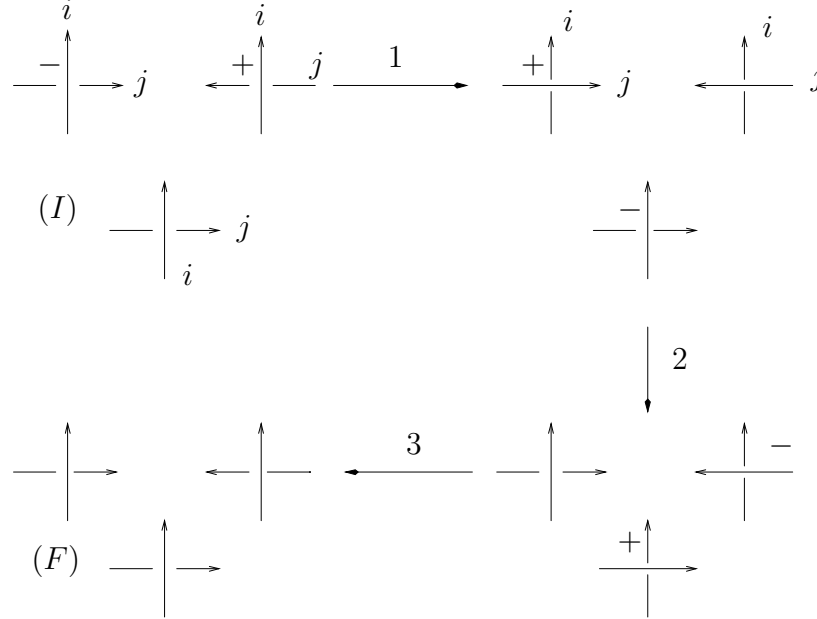
3.1.1. *2-T Relation.* In the following picture we can move the loop circling the double point upward, using double crossing flips.



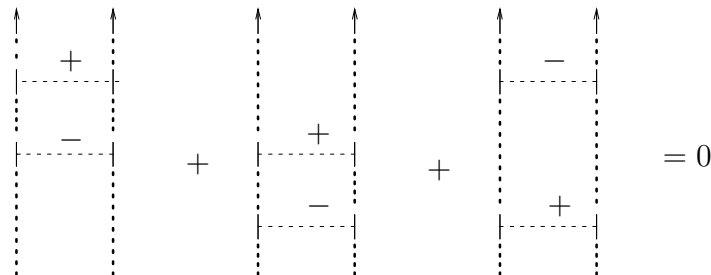
Change crossings 1 and 3 simultaneously and then 2 and 4. The result now will be the 2-T relation.



3.1.2. *3-T Relation.* Consider the following sequence of moves involving the i^{th} and j^{th} components of the link.



As a result of applying this sequence, the initial and final configurations (I) and (F) are the same. Then in terms of the diagrams this sequence translates into the fact that each pair of chords created while performing each step can be rewritten in terms of the other two pairs.



As an immediate consequence of the 3-T relation we obtain a 1-T relation, which also can be proven using the second Reidemeister move.

$$= 0$$

3.2. \bigwedge Diagrams. First note that the 2-T, 3-T and their consequences presented earlier are also valid for wedge diagrams. the following two figures show the appropriate setting of those relations for wedge diagrams.

3.2.1. \bigwedge -2-T. for wedge diagrams the 2-T relation becomes,

$$= 0$$

Notice the change of components where the wedge type intersection occurs.

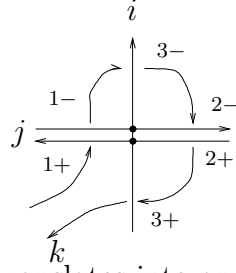
3.2.2. \bigwedge -3-t.

$$= 0$$

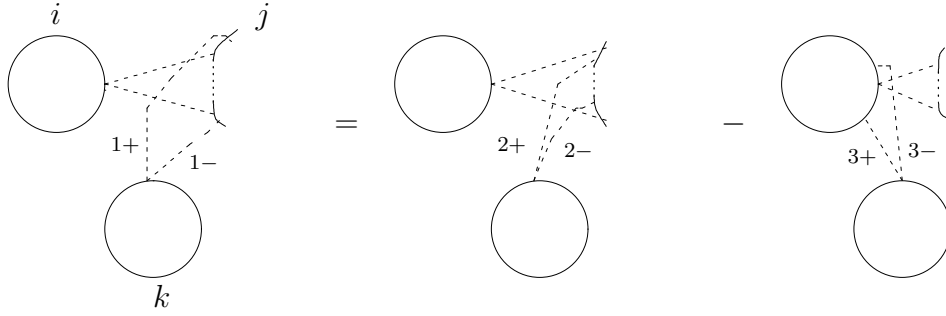
3.2.3. Interchange Relation. The situation described in the next figure can be treated as in the 2-T relation, only that here we must assure that the crossings we change are close.

This can be done in two inequivalent ways, each corresponds to a certain pairing of the crossings.

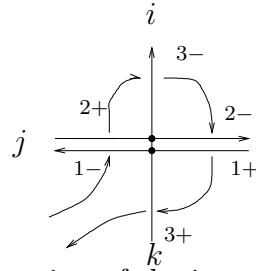
1.



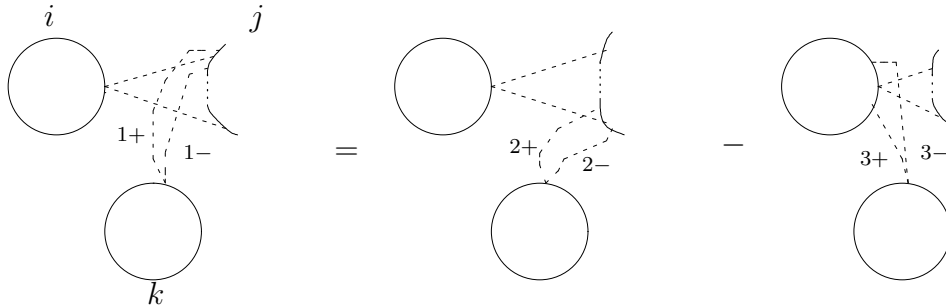
In terms of wedge diagrams, this translates into an interchange relation.



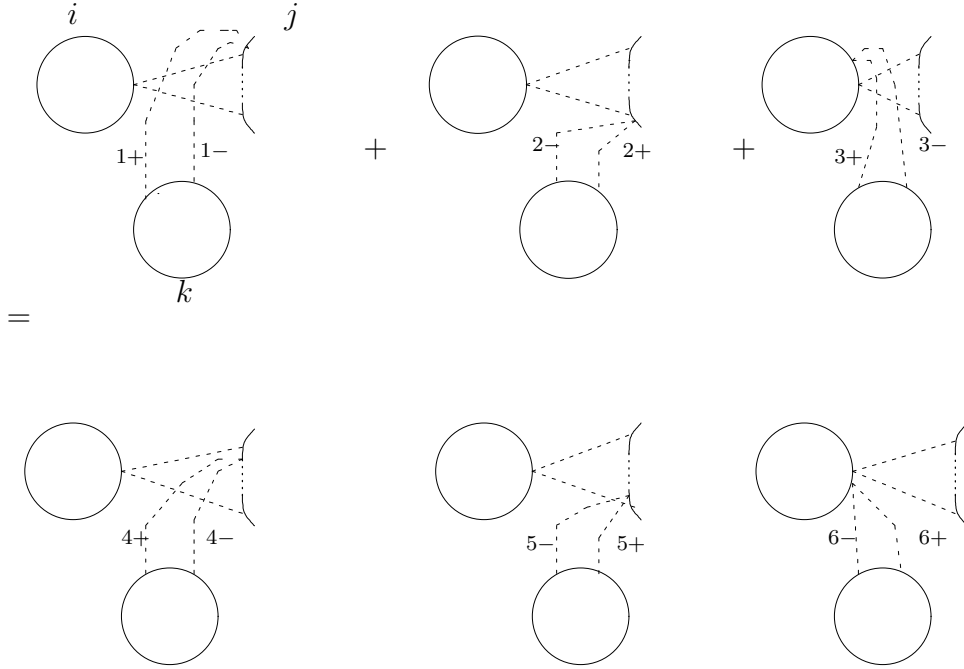
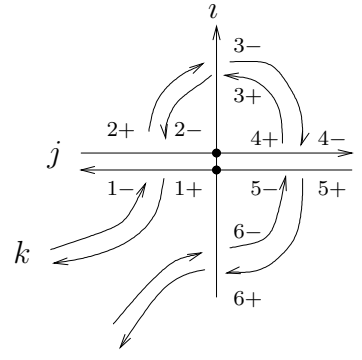
2.



In this case we get the following version of the interchange relation.

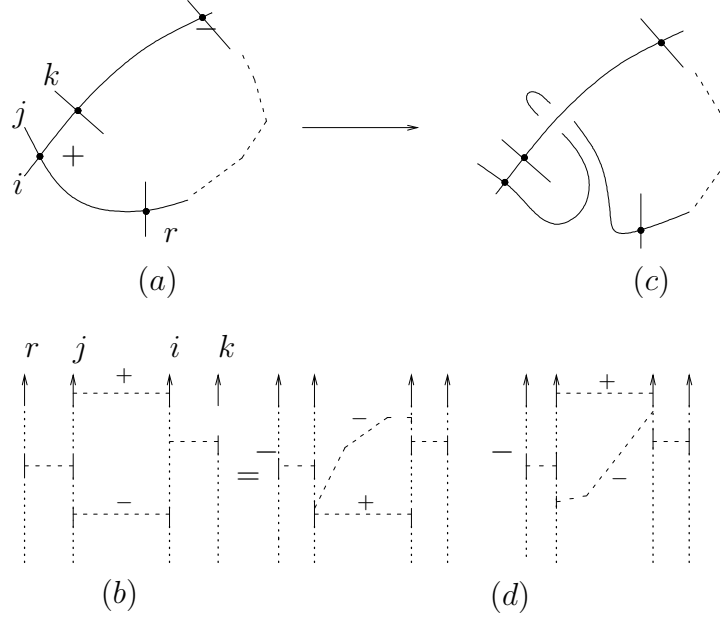


3.2.4. *Wood Pecker relation.* Now consider a case where the two pieces of the k^{th} component loop around a double singularity as shown on the right. Again move the loops upward, performing the necessary double crossing changes and follow them by the matching wedge diagrams. You will get the Wood Pecker relation shown below:



As said earlier in section 2 Wedge diagrams naturally included in Double Dating diagrams. Since diagrams are the dual space of the space of invariants and since every DD-finite type invariant is a \bigwedge -finite type invariant this map between the spaces of diagrams should be

onto. Indeed it is. The topological proof of this fact is illustrated in the following.



In the above left (a) we have a pair of intersection points between the components i and j signed $+/ -$, and divided by intersections with the k and r components. (b) is the double dating diagram corresponding to it. We use a finger move (c) and the 3-T relation in order to have (b), as a sum of diagrams (d) in which one of them is a wedge diagram and the other is “almost” wedge. Applying this procedure as much as necessary makes the double dating diagram (b) become a sum of wedge diagrams.

4. SOME LOW DEGREE COMPUTATIONS

Denote the vector space of all type n invariants of links with m components, and linking matrix P , by $\mathcal{F}_{P,m}^n$.

Theorem 4.1. *The dimensions of $\mathcal{F}_{P,m}^m$ for $n = \{0, 1, 2\}$ are given by,*

$$\dim \mathcal{F}_{P,m}^0 = 1 \text{ for all } m.$$

$$\dim \mathcal{F}_{P,m}^1 = 0 \text{ for all } m.$$

$$\dim \mathcal{F}_{P,m}^2 = m \text{ for all } m \text{ where } P \neq 0.$$

$$\dim \mathcal{F}_{0,3}^2 = 3 + 1.$$

$$\dim \mathcal{F}_{0,m}^2 = m + \binom{m}{3} \text{ for all } m > 3.$$

Proof. The invariants of type 0 are the constants. There is a unique Vassiliev invariant of type 2 for knots, which is V_2 (the second coefficient of the Conway invariant). Therefore for a link with m component, we have m V_2 's each one for each component of the link. The extra invariant in $\mathcal{F}_{0,3}^2$ is the Milnor μ_{ijk} . If $m > 3$ then we also have $\binom{m}{3}$ such invariants. The list of finite type invariants given above, consists of the invariant of type ≤ 2 which come from Vassiliev invariant of links and tangles. This list gives lower bounds on the dimensions claimed in the theorem. The completion of the proof is done by calculating the dimensions of the space of double dating diagrams up to degree 2. This calculation is the content of the next theorem. \square

Let $\mathcal{A}_m^{DD,n}$ be $\mathcal{D}_m^{DD,n}$ modulo 1-T, 2-T, 3-T.

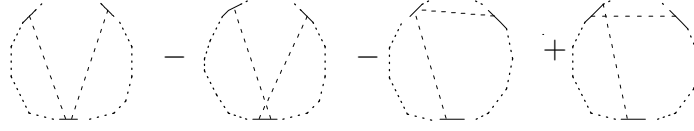


Figure 9. 4-T relation

Theorem 4.2. *The dimensions of $\mathcal{A}_m^{DD,n}$ up to degree 2 are given by,*
 $\dim \mathcal{A}_m^{DD,0} = 1$ *for all* m .
 $\dim \mathcal{A}_m^{DD,1} = 0$ *for all* m .
 $\dim \mathcal{A}_m^{DD,2} = \binom{m}{3}$

Proof. The proof of the theorem is done by using the 1-T, 2-T and 3-T which give sequences of equalities between double dating diagrams. Those equalities are given in Appendix A. \square

5. FROM DOUBLE DATING DIAGRAMS TO CHORD DIAGRAMS

We briefly recall the definitions of the algebra \mathcal{A}^c of chord diagrams, and \mathcal{A}^t of trivalent diagrams. See [BN1] for further details. A chord diagram is a disjoint union of oriented circles with dashed chords attached to them.

$$\mathcal{A}^c = \text{span}\{\text{chord diagrams}\}/4\text{-T relation}$$

A trivalent diagram is the same as chord diagram with the exception that there are internal vertices allowed, where an internal vertex is a vertex where three chords meet. Every internal vertex is oriented by ordering the three edges meeting at this vertex. \mathcal{A}^t is the algebra generated by trivalent diagrams modulo the previous 4-T, plus the additional AS and STU relations.

STU

$$\text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} = 0$$

$$\text{Diagram 4} + \text{Diagram 5} = 0$$

AS

In [BN1] it is shown that \mathcal{A}^c and \mathcal{A}^t are isomorphic. Recall from Theorem 1.5 that Vassiliev invariants are naturally mapped into finite type invariants under the inclusion map. A natural question arising from this observation is whether or not this map is an 1 : 1 and is it onto. In this section we will introduce a map from the space \mathcal{A}^{DD} into \mathcal{A}^c which is the natural dual map to the inclusion map from Vassiliev invariants into finite type invariants. We shall find the image under this map of the space \mathcal{A}^\wedge of wedge diagrams and since the inclusion map from wedge diagram to double dating diagram was shown to be onto, we can identify also the image of \mathcal{A}^{DD} .

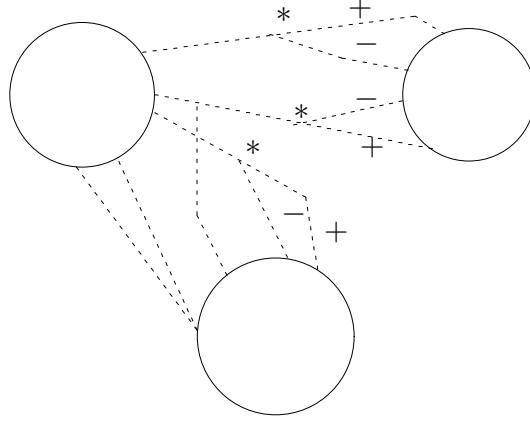


Figure 10. A diagram in $\mathcal{D}^{\Lambda, t}$

Definition 5.1. Define $\iota : \mathcal{D}^{DD} \rightarrow \mathcal{D}^c$ by defining

$$\iota \left(\begin{array}{c} \uparrow \quad + \quad \uparrow \\ \text{---} \\ \uparrow \quad - \quad \uparrow \\ \text{---} \end{array} \right) = \begin{array}{c} \uparrow \quad \uparrow \\ \text{---} \\ \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \\ \text{---} \\ \uparrow \quad \uparrow \end{array}$$

On wedge diagrams the restriction of ι will look as

$$\iota \left(\begin{array}{c} \text{---} \\ \text{---} \\ \uparrow \quad + \quad \uparrow \\ \text{---} \\ \uparrow \quad - \quad \uparrow \\ \text{---} \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \text{---} \\ \uparrow \quad \uparrow \\ \text{---} \\ \uparrow \quad \uparrow \\ \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \text{---} \\ \uparrow \quad \uparrow \\ \text{---} \\ \uparrow \quad \uparrow \\ \text{---} \\ \text{---} \end{array}$$

In general, ι of a diagram is defined by taking the sum over all pairs of chords.

It is easily checked that ι is well defined map from \mathcal{A}^{DD} to \mathcal{A}^c . This is done by showing that all the relations in \mathcal{A}^{DD} are mapped into the subspace of \mathcal{A}^c spanned by all $4 - T$ relations. This is done in appendix B. We view ι as a map from \mathcal{A}^{DD} into \mathcal{A}^t

Theorem 5.2. The image of \mathcal{A}^Λ under ι is the subspace, \mathcal{A}^{st} , of \mathcal{A}^t , consisting of strutless diagrams in which there is no chord having both its ends as external vertices.

Proof. In order to prove the theorem we introduce another space $\mathcal{D}^{\Lambda, t}$ having \mathcal{D}^Λ as a subspace. Further we will define a map $\bar{\iota}$ from $\mathcal{D}^{\Lambda, t}$ into \mathcal{A}^t which extends ι and we will show that $\mathcal{D}^{\Lambda, t}$ is mapped into the subspace of strutless diagrams. The objects of $\mathcal{D}^{\Lambda, t}$ are diagrams having trivalent vertices some of which external where a chord meets a circle component of the diagram, and some are internal where three chords meet. Some of the internal vertices are marked by *. The chords that meet at an internal marked vertex are required that at least two of them are attached to the same circle-component of the diagram. those two chords are also signed by +, - signs. In Figure 10 there is an example of a diagram in $\mathcal{D}^{\Lambda, t}$. Now define $\bar{\iota}$ by

The diagram shows an equality between two expressions. On the left, a circle labeled $\bar{\iota}$ is connected by a dashed line to a trivalent vertex marked with an asterisk (*). This vertex has two outgoing dashed lines, one labeled '+' and the other labeled '-'. On the right, the same expression is equal to a negative sign followed by a sum of two diagrams. Each diagram in the sum consists of a circle connected to a trivalent vertex, which then branches into two dashed lines, one labeled '+' and the other labeled '-'.

and if a diagram contains trivalent vertices non of them is a marked vertex, then it is mapped to zero. Notice that the following holds.

The diagram shows an equality between two expressions. On the left, a circle is connected by a dashed line to a trivalent vertex marked with an asterisk (*). This vertex has two outgoing dashed lines, one labeled '+' and the other labeled '-'. On the right, the same expression is equal to a sum of two diagrams. Each diagram in the sum consists of a circle connected to a trivalent vertex, which then branches into two dashed lines, one labeled '+' and the other labeled '-'.

Suppose now that there is a marked vertex for which there are other strands touching the circle where the two end points are, on the arc between those two points. Then after applying $\bar{\iota}$ as much as necessary, the following equality.

The diagram shows an equality between two expressions. On the left, a circle is connected by a dashed line to a trivalent vertex marked with an asterisk (*). This vertex has two outgoing dashed lines, one labeled '+' and the other labeled '-'. On the right, the same expression is equal to a sum of three diagrams. Each diagram in the sum consists of a circle connected to a trivalent vertex, which then branches into two dashed lines, one labeled '+' and the other labeled '-'. A label 'legs of incoming chords' points to the dashed lines in the top diagram.

Hence the only thing to worry about is an object as in the right hand side of the previous equation. The claim is finally proven by the two following equalities.

The diagram shows an equality between two expressions. On the left, a circle is connected by a dashed line to a trivalent vertex marked with an asterisk (*). This vertex has two outgoing dashed lines, one labeled '+' and the other labeled '-'. A label 'a chord' points to the dashed line. On the right, the same expression is equal to a sum of two diagrams. Each diagram in the sum consists of a circle connected to a trivalent vertex, which then branches into two dashed lines, one labeled '+' and the other labeled '-'.

The diagram shows an equality between two expressions. On the left, a circle is connected by a dashed line to a trivalent vertex marked with an asterisk (*). This vertex has two outgoing dashed lines, one labeled '+' and the other labeled '-'. On the right, the same expression is equal to a sum of two diagrams. Each diagram in the sum consists of a circle connected to a trivalent vertex, which then branches into two dashed lines, one labeled '+' and the other labeled '-'.

On the other hand, the fact that $\bar{\iota}$ is onto \mathcal{A}^{st} follows from the following equality.

$$\text{Y-junction with dashed lines} = \bar{\iota} \left(\text{Y-junction with dashed lines} \right)$$

Using STU we have that every strutless diagram is a sum of diagrams in which every trivalent vertex is adjacent to a circle as in the figure, and each such a diagram is in the image of ι \square

5.1. The Case of Knots. Let us focus on invariants of framed knots with a fixed self linking number. Both double dating and chord diagrams in this case are made of a single circle with dashed chords attached, and we have the appropriate relations in each type of diagrams. Define a map ν from \mathcal{A}^c into \mathcal{A}^{DD} by setting,

$$\nu \left(\text{circle with dashed lines} \right) = \text{circle with dashed lines and isolated chord}$$

To each chord in the diagram add an isolated chord somewhere in the figure and sign them as indicated

Applying ν on the 4-T relation yields the following two equalities that assures that ν is a well defined map from \mathcal{A}^c into \mathcal{A}^{DD} .

$$\text{Diagram 1} - \text{Diagram 2} = \text{Diagram 3} - \text{Diagram 4} = 0$$

$$\text{Diagram 5} - \text{Diagram 6} = \text{Diagram 7} = 0$$

Notice that if D is a chord diagram then the following holds.

$$\iota \circ \nu(D) = D + \text{sum of diagrams each of which has an isolated chord}$$

As a corollary of the previous equality, we have that in the case of unframed knots the restriction of ι to \mathcal{A}^{DD} is a 1-1 map into $\mathcal{A}^{c,r}$ which is the same as \mathcal{A}^c divided by the framing independence relation which says that a chord diagram has an isolated chord then it is equal to zero. From the definition of ν it is clear that ι is onto $\mathcal{A}^{c,r}$. The meaning of the corollary is that in the case of unframed knots, the theory of finite type invariant for unframed knots with a fixed self linking number is the same as the theory of Vassiliev invariants of framed knots.

5.2. 2-Component String Links. Recall [BN2] for the definition of a string link with m components. In the case of 2-components string tangles we can construct a map ν with the same behaviour as for knots. For every chord in a chord diagram D , we add another chord,

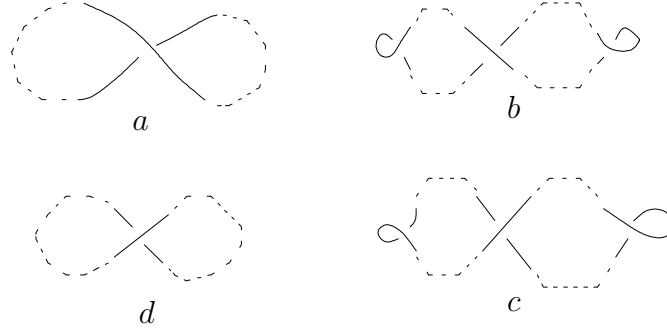


Figure 11. A double flip(b to c) representing a single flip (a to d)

at the bottom of the diagram and we put + sign on the original chord, and a - sign on the added new chord. See the following example.

$$\nu \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) = \begin{array}{c} + \\ 1+ \\ 2+ \\ 1- \\ 2- \\ - \end{array}$$

Using the relations as before shows that this map is well-defined. As a corollary we have, for two components tangles as well, that both theories, of Vassiliev and finite type invariants are the same.

5.2.1. Remark. If we try to define a map similar to ν also for m -string tangles where $m > 2$, this will not do, since the map will not be well-defined. The order in which we add the negative chords is not irrelevant in this case. We can interpret these corollaries geometrically as follows. If we have a singular knot and a given intersection point on it, we can artificially create two kinks, having signs opposite to each other and as a result one of them has a sign opposite to the crossing we wish to change when we resolve the given intersection point. Performing a double-crossing change in which, the flipped crossings are the crossing given by the intersection point and the crossing which creates the kink with the opposite sign, is topologically equivalent to performing the crossing flip of the intersection point. The self linking number remains fixed during this procedure, but still we should bear in mind the fact that the framing is changed. This is indicated in figure 11. For 2-string tangles we have a similar way of presenting a single-crossing change using a double-crossing change. Instead of kinks we create a trivial pair of crossings for each crossing we wish to flip.

5.2.2. The invariant μ_{ijk} . Calculating ι on the generator of the space of diagrams of degree 2 with 3 components gives,

$$\iota \left(\begin{array}{c} i \quad j \\ \circ \quad \circ \\ \circ \end{array} \right) = \begin{array}{c} i \quad j \\ \circ \quad \circ \\ \circ \end{array}$$

By [HaLi] the coefficient of the Chinese character obtained from this diagram by removing the circles, in a certain algebra structure, gives the weight system of the Milnor invariant μ_{ijk} .

6. OPEN QUESTIONS

We will now present some questions yet left open.

1. Are there any other relations in the spaces $\mathcal{D}^{DD}, \mathcal{D}^\Lambda$ not presented here ?
2. What are the dimensions of $\mathcal{A}^{DD}, \mathcal{A}^\Lambda$ in degrees higher than 2 ?
3. What is the kernel of *iota* ?
4. How should the map ν from \mathcal{A}^c to $\mathcal{A}^{DD}, \mathcal{A}^\Lambda$ be defined in the general case, can it be well defined ?

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APPENDIX A. GUIDED TOUR THROUGH DOUBLE DATING DIAGRAMS

In this section we use the relations in order to divide all DD diagrams into their equivalence classes. Each of the equalities listed here should be understood as equivalence modulo the relations given in section 3. Namely, two diagrams are equivalent if the difference between them lies in the subspace spanned by all the relations. For each equality we specify the relation by which the equality is derived. When we say that some diagram, D , is equal to zero we mean that either it has an embedding K_D on which all finite type invariant vanish or that D is equivalent, via the relations to such a diagram. In the following we will refer to such k_D as a geometric representative. All equalities can be interpreted geometrically using appropriate embeddings and drawing the crossing changes insinuated by the relations. We

2. One isolated pair

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = 0 \\
 & \text{Diagram 3} = \text{Diagram 4} = \text{Diagram 5} = \text{Diagram 6} = 0
 \end{aligned}$$

$3 - T$

3. Two non-isolated pairs

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = 0 \\
 & \text{Diagram 3} = \text{Diagram 4} = 0 \\
 & \text{Diagram 5} = \text{Diagram 6} = \text{Diagram 7} = \text{Diagram 8} = 0 \\
 & \text{Diagram 9} = - \text{Diagram 10} = 1
 \end{aligned}$$

using 2-T *existing of a geometric representative* *180 degrees rotation*

Now one can choose a representative for equivalence class and have the result of theorem 4.2

APPENDIX B. CALCULATING IOTA

In this section we present the results obtained after applying ι to the relations we have for DD and \bigwedge diagrams.

$$\begin{aligned}
 \iota(2-T) &= \text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4} \\
 &+ \text{Diagram 5} - \text{Diagram 6} + \text{Diagram 7} - \text{Diagram 8} = 4-T
 \end{aligned}$$

Applying ι on the wood pecker relation gives the following sum. Applying ι once more will yield a sum of two 4-Ts and some terms that vanish in pairs.

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